UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 3rd Semester Examination, 2020

CC7-MATHEMATICS

RIEMANN INTEGRATION AND SERIES OF FUNCTIONS

Full Marks: 60

ASSIGNMENT

The figures in the margin indicate full marks.
All symbols are of usual significance.

Answer *all* questions

GROUP-A

1. Answer ***all*** questions: 2×6 = 12

(a) Compute $L(P, f)$ and $U(P, f)$ if $f(x)=x^2$ on [0, 1] and $P=\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$.

(b) Show by an example that every bounded function need not be Riemann integrable.

(c) If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges for all real x, prove that $\lim_{n\to\infty} |a_n|^{1/n}=0$.

(d) Prove that the series $\sum_{n=0}^{\infty} \frac{1}{n^3+n^4x^2}$ is uniformly convergent for all real x.

(e) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ where $a_n=2^n+3^n$, $n\geq 1$.

(f) Give an example of function f and g both integrable on $[a, b]$ such that

$$\int_a^b |f-g|=0 \text{ but } f\neq g$$

GROUP-B

Answer *all* questions 5×4 = 20

2. Let $f_n(x)=\log(n^2+x^2)$, $x\in\mathbb{R}$. Show that the sequence $\{f_n'\}$ is uniformly convergent on $\mathbb{R}$ but $\{f_n\}$ is not uniformly convergent on $\mathbb{R}$. 5

3. Show that for the function f defined on $0 \le x \le 1$ as 5

$$f(x) = \sqrt{1-x^2} \quad , \; x \text{ is rational}$$
$$= 1-x \quad , \; x \text{ is irrational}$$

$\underline{\int_a^b} f(x)\,dx = \frac{1}{2}$ and $\overline{\int_a^b} f(x)\,dx = \frac{\pi}{4}$ and so $f(x)$ is not integrable on [0, 1].

4. A function f is defined on $\left(-\frac{1}{3}, \frac{1}{3}\right)$ by 5

$$f(x) = 1 + 2.3x + 3.3^2x^2 + \ldots\ldots + n.3^{n-1}x^{n-1} + \ldots\ldots$$

Show that f is continuous on $\left(-\frac{1}{3}, \frac{1}{3}\right)$. Evaluate $\int_{1/4}^{0} f$.

5. (a) Let $f:[a, b] \longrightarrow R$ be function on $[a, b]$. Show that if f is integrable on $[a, b]$ then $|f|$ is integrable on $[a, b]$, but the converse is not true. 4

(b) Let $f_n(x) = \tan^{-1} nx$, $x \in [0, 1]$. Prove that the sequence $\{f_n\}$ is not uniformly convergent on [0, 1]. 1

GROUP-C

Answer *all* questions 7×4 = 28

6. (a) Let $R(>0)$ be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$. Prove that the radius of convergence of the series obtained by integrating $\sum_{n=0}^{\infty} a_n x^n$ term-by-term is also R. 3

(b) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R(>0)$ and $f(x)$ be the sum of the series on $(-R, R)$. Show that $f^k(0) = k!\, a_k \; (k = 0, 1, 2, \ldots..)$. 4

7. (a) Construct a sequence of functions $\{f_n\}_{n\in \mathbb{N}}$ on [0, 1] such that each f_n is R-integrable on [0, 1], $\{f_n\}_{n\in \mathbb{N}}$ converges pointwise on [0, 1] to f and f is not R-integrable on [0, 1]. 4

(b) Show that the sequence $\{f_n\}_{n\in \mathbb{N}}$ where $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ is uniformly convergent on $[0, \pi]$. 3

8. (a) Prove that a bounded real valued function $f:[a, b]\longrightarrow \mathbb{R}$ is Riemann integrable on $[a, b]$ if and only if there exist a partition P of $[a, b]$ such that $U(P, f)-L(P, f)<\varepsilon$. Is this result true for any replacement of P? Is this result true for unbounded function? — Justify. 5

(b) Let $f:[a, b]\longrightarrow \mathbb{R}$ be bounded function. Suppose that there is a partition P of $[a, b]$ such that $L(P, f)=U(P, f)$. Show that f is a constant function. 2

9. (a) If a sequence of functions $\{f_n\}_{n\in \mathbb{N}}$ converges uniformly on $[a, b]$ to a function f and if $c\in [a, b]$ s.t. $\lim\limits_{x\to c} f_n(x)=a_n$, $n\in \mathbb{N}$. Show that 5

(i) $\{a_n\}_{n\in \mathbb{N}}$ converges

(ii) $\lim\limits_{x\to c} f(x)$ exists

(iii) $\lim\limits_{x\to c}\lim\limits_{n\to\infty} f_n(x)=\lim\limits_{n\to\infty}\lim\limits_{x\to c} f_n(x)$.

Deduce further that if each f_n be continuous on $[a, b]$, then the limit function f is continuous on $[a, b]$.

(b) Give an example to show that the continuity of $f(x)$ is not a necessary condition for the existence of an antiderivative of $f(x)$. 2

——×——