UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 3rd Semester Examination, 2020

CC5-MATHEMATICS

THEORY OF REAL FUNCTIONS AND INTRODUCTION TO METRIC SPACES

Full Marks: 60

ASSIGNMENT

The figures in the margin indicate full marks.
All symbols are of usual significance.

Answer *all* questions

GROUP-A

1. Answer ***all*** questions: 2×6 = 12

(a) Let f be a real valued function defined over [–1, 1] such that 2

$$f(x)=\begin{cases} x\cos\frac{1}{x} & , \quad \text{when} \quad x\neq 0 \\ 0 & , \quad \text{when} \quad x=0 \end{cases}$$

Does the Mean Value Theorem hold?

(b) If $a>0,\ b>0$, then find $\lim\limits_{x\to 0^+}\left[\frac{x}{a}\right]\frac{b}{x}$. 2

(c) Obtain a relation between a and b so that $\lim\limits_{x\to 0}\dfrac{a\sin 2x-b\sin x}{x^2}=1$. 2

(d) Prove that a subset A of a metric space $(X,\,d)$ is a singleton set iff $\delta(A)=0$. 2

(e) Let $H=\{\frac{1}{2^p}+\frac{1}{3^q}\ ;\ \ p,\,q\in\mathbb{N}\}$. Then obtain (i) derived set H' of H (ii) derived set $(H')'$ of H'. 2

(f) Let $f(x)=|x-1|+|x-2|,\ \ x\in[0,\,3]$, show that 2 is a local minimum of f. 2

GROUP-B

2. Answer ***all*** questions: 5×4 = 20

(a) Evaluate $\lim\limits_{x\to 0}\dfrac{\tan x-x}{x-\sin x}$. 5

(b) Show that the function f on [0, 1] defined as 5

$$f(x)=\begin{cases}\frac{1}{2^n}, & \frac{1}{2^{n+1}}<x\le\frac{1}{2^n}\ ;\ n=0,1,2,\ldots.\\ 0, & x=0\end{cases}$$

is discontinuous at $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\ldots$

(c) Prove that any two real roots of the equation $e^x\cos x+1=0$ there is at least one real root of the equation $e^x\sin x+1=0$. 5

(d) Let M denote the set of all bounded sequences of real numbers. If $x=\{x_n\}_{n=1}^{\infty}$ and $y=\{y_n\}_{n=1}^{\infty}$ are points of M, then prove that the function 5

$$d(x, y)=\underset{1\le n<\infty}{\text{lub}}\ |x_n-y_n|$$

is a metric on M.

GROUP-C

3. Answer ***all*** questions: 7×4 = 28

(a) (i) Show that the volume of the greatest cylinders which can be inscribed in a cone of height h and semi-vertical angle α in $(4/27)\pi h^3\tan^2\alpha$. 4+3

(ii) A function $f:\mathbb{R}\longrightarrow\mathbb{R}$ is continuous function and $f(x)=0$ for all $x\in\mathbb{Q}$. Prove that $f(x)=0$ for all $x\in\mathbb{R}$.

(b) (i) Prove that $\dfrac{x}{1+x^2}<\tan^{-1}x<x$, if $x>0$. 3+4

(ii) Let A be a non-empty subset of $\mathbb{R}$. A function $f:\mathbb{R}\longrightarrow\mathbb{R}$ is defined by $f_A(x)=\inf\{|x-a|: a\in A\}$. Prove that f_A is uniformly continuous on $\mathbb{R}$.

(c) (i) Show that for all $x, y\in\mathbb{R}$, $d(x, y)=|\tan^{-1}x-\tan^{-1}y|$ is a metric on $\mathbb{R}$, which is bounded too. 4+3

(ii) Show that the sets $A=\mathbb{N}$, $B=\left\{n+\dfrac{1}{2n}: n\in\mathbb{N}\right\}$ in $\mathbb{R}$ are closed and disjoint. What is $d(A, B)$?

(d) (i) For each $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)\in\mathbb{R}^n$, show that 3+4

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2\le\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right)$$

(ii) If (X, d) be a metric space. Then for each $x\in X$ and for each $\varepsilon>0$, show that $\{y\in X\,|\,d(x, y)<\varepsilon\}$ is an open subset and $\{y\in X\,|\,d(x, y)\le\varepsilon\}$ is a closed subset of X with respect to d.

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