UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 1st Semester Examination, 2020

GE-MATHEMATICS

Full Marks: 60

ASSIGNMENT

The figures in the margin indicate full marks.
All symbols are of usual significance.

The question paper contains GE1, GE2, GE3, GE4 and GE5. Candidates are required to answer any *one* from the *five* courses and they should mention it clearly on the Answer Book.

GE1

CALCULUS, GEOMETRY, AND DIFFERENTIAL EQUATION

GROUP-A

1. Answer ***all*** the questions from the following: 2×6 = 12

(a) Evaluate: $\lim\limits_{x\to\infty}\left(\sqrt{x^2+2x}-x\right)$

(b) If $y=\dfrac{1}{x^2+a^2}$, find y_n where suffix n represents n-th derivative.

(c) Evaluate: $\displaystyle\int_0^{\pi/4}\tan^6 x\,dx$

(d) Find the point of inflexion on the curve $x=(y-1)(y-2)(y-3)$.

(e) Find the rectilinear asymptotes to the curve $y=xe^{1/x}$.

(f) Find the volume of the solid obtained by revolving the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ about its axis of symmetry.

GROUP-B

2. Answer ***all*** the questions from the following: 5×4 = 20

(a) Find the envelope of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, where the parameters are connected by the relation $a^n+b^n=c^n$ (c being fixed).

(b) If $y^{1/m}+y^{-1/m}=2x$, prove that

$$(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$$

(c) Find the area bounded by one arch of the cycloid $x=(\theta-\sin\theta)$, $y=a(1-\cos\theta)$ and the x-axis.

(d) If $I_n=\int\frac{\sin nx}{\sin x}\,dx$, show that $(n-1)(I_n-I_{n-2})=2\sin(n-1)x$.

GROUP-C

3. Answer ***all*** the questions from the following: 7×4 = 28

(a) (i) Find the asymptotes of $(x^2-a^2)y^2=x^2(x^2-4a^2)$. 3

(ii) Prove: $\lim\limits_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}=e^{\frac{1}{3}}$ 4

(b) (i) Find the volume of the solid generated by revolving the cardioid $r=a(1+\cos\theta)$ about the initial line. 3

(ii) Trace the curve $x^3+y^3=3axy$. 4

(c) (i) Show that the curve $y=\frac{(1-x)}{(1+x^2)}$ has three points of inflexion which lie on a straight line. 3

(ii) Show that the envelope of the family of circles whose centres lie on the rectangular hyperbola $xy=c^2$ and which pass through the centre of the hyperbola is $(x^2+y^2)^2=16c^2xy$. 4

(d) (i) Find the area enclosed between the curve $y=x^3$ and the line $y=x$. 3

(ii) If $I_n=\int_0^1 x^n\tan^{-1}x\,dx$ $(n>2)$, then prove that $(n+1)I_n+(n-1)I_{n-2}=\frac{\pi}{2}-\frac{1}{n}$. 4

GE2

ALGEBRA

GROUP-A

1. Answer ***all*** the questions from the following: 2×6 = 12

(a) If the roots of the equation $x^3+px^2+qx+r=0$ are in Arithmetic Progression, prove that $p^2\geq 3q$. 2

(b) Check whether the matrix $\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & -1\\ 5 & 7 & 3\end{pmatrix}$ is invertible. 2

(c) Let $T:\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be a linear transformation defined by $T(x,\ y,\ z)=(2x-y,\ \ y+z)$. 2

Find the matrix representation of T relative to the basis $\{(1,0,0),(0,1,0),(0,0,1)\}$.

(d) Check whether the set $\{v_1=(1,2,3),\ v_2=(4,5,6),\ v_3=(2,1,0)\}$ is linearly independent. 2

(e) Apply Descartes' rule of signs to find the nature of the roots of the equation $x^4+qx^2+rx-s=0$ $(q,\ r,\ s\in \mathbb{Z}^+)$. 2

(f) Find the rank of the linear transformation $T:\ \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(x,\ y,\ z)=(y,\ z,\ 0)$. 2

GROUP-B

Answer *all* the questions from the following 5×4 = 20

2. Solve the equation $x^5-1=0$ and hence deduce the values of $\cos\frac{\pi}{5}$ and $\cos\frac{2\pi}{5}$. 5

3. Determine the values of a and b for which the system 5

$$x+y+z=1$$
$$x+2y-z=b$$
$$5x+7y+az=b^2$$

has no solution.

4. Solve the equation $x^4-2x^2+8x-3=0$ by Ferrari's method. 5

5. If α,β,γ be the roots of the equation $x^3+px^2+qx+r=0$, find the value of $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$. 5

GROUP-C

Answer *all* the questions from the following 7×4= 28

6. (a) Solve by Cardan's method: 4

$$x^3-3x+3=0$$

(b) If α,β,γ are the roots of the equation $x^3+2x^2+1=0$, find the equation whose roots are $\alpha+\frac{1}{\alpha}$, $\beta+\frac{1}{\beta}$, $\gamma+\frac{1}{\gamma}$. 3

7. (a) Determine the special roots of the equation $x^6-1=0$. 4

(b) Solve if possible the system of equations: 3

$$x+2y+3z=1$$
$$2x+4y+2z=-1$$
$$x+2y-2z=5$$

8. (a) Transform the following matrix into row reduced echelon matrix: 4

$$\begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}$$

(b) Define a linear transformation $T:\ \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by 3

$$T(x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad \forall\ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$$

Find the images of $u = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $v = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $u + v = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$ under *T*.

9. (a) If the sum of two roots of the biquadratic equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ is zero, find the roots. 4

(b) Find the inverse of $A = \begin{pmatrix} 0 & -9 & 1 \\ 5 & 1 & 6 \\ -2 & 1 & 4 \end{pmatrix}$. 3

GE3

DIFFERENTIAL EQUATION AND VECTOR CALCULUS

GROUP-A

1. Answer the following questions: 2×6 = 12

(a) Solve: $(x^2y - 2xy^2)\ dx + (3x^2y - x^3)\ dy = 0$

(b) Solve: $x\sqrt{y}\ dx + (1+y)\sqrt{(1+x)}\ dy = 0$

(c) Find the Wronskian of $[1,\ x,\ x^2]$.

(d) Construct the differential equation of all circle each of which touches the axis of *x* at origin.

(e) Solve:

$$\frac{dx}{dt} = -\omega y \quad ; \qquad \frac{dy}{dt} = \omega x$$

(f) Solve: $\dfrac{x\,dx + y\,dy}{x^2 + y^2}$

GROUP-B

2. Answer the following questions: 5×4 = 20

(a) Solve: $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 2\log x$

(b) Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2}+a^2y=\sec ax$$

(c) Solve by using the method of undetermined coefficients:

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=e^{2x}$$

(d) Solve: $\frac{dy}{dx}+y\cos x=y^n\sin 2x$

GROUP-C

3. Answer the following questions: 7×4 = 28

(a) (i) Show that $x^{-1}\sin x$ is a part of the complementary function for the differential equation 3

$$\frac{d^2y}{dx^2}+(1+\frac{2}{x}\cot x-\frac{2}{x^2})y=x\cos x$$

(ii) Solve: 4

$$\frac{dx}{dt}+5x-2y=e^t$$

$$\frac{dy}{dt}-x+6y=e^{2t}$$

(b) (i) Solve: 3

$$\frac{d^2y}{dx^2}=\sec^2 y\,\tan y$$

(ii) Solve: $x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=\frac{1}{(1-x)^2}$ 4

(c) (i) Solve: 3

$$\frac{dy}{dx}=\frac{\sin x+x\cos x}{y\,(2\log y+1)}$$

(ii) Find the Lipschitz constant for the function $f(t,\ y)=6ty^{2/3}$. 4

(d) (i) Find the value of u which satisfies the equation 4

$$\frac{d^2u}{d\theta^2}+u=2k\cos\theta$$

and also the following conditions:

u has the same value when $\theta=\pm\,\pi/2$ and $\int_0^{\pi/2}u\,d\theta=0$.

(ii) Solve: $(1+3e^{x/y})\,dx+3e^{x/y}(1-x/y)\,dy=0$ 3

GE4

GROUP THEORY

GROUP-A

1. Answer ***all*** the questions from the following: 2×6= 12

(a) If G be a commutative group, then prove that $H=\{a^2 : a \in G\}$ is a subgroup of G. 2

(b) Let a, b, c be three elements of a group G. Show that 2

$$(abc)^{-1}=c^{-1}b^{-1}a^{-1}$$

(c) What is the inverse permutation of (1, 5) (2, 3, 4)? 2

(d) Give an example of a finite subgroup of an infinite group. 2

(e) Write down the elements in the group S_3. 2

(f) Show that the identity element in a subgroup of a group is the same as that of the group. 2

GROUP-B

Answer *all* the questions from the following 5×4 = 20

2. Find the eight symmetries of a square and show that they form a non-abelian group. 5

3. Let $M:=\left\{\begin{pmatrix} x & y \\ x & y \end{pmatrix} ;\ x,\ y \in \mathbb{R},\ \ x+y \neq 0\right\}$ 5

Check whether M forms a group with respect to matrix multiplication.

4. Let $GL_2(\mathbb{R})$ be the group of invertible 2×2 real matrices with respect to the matrix multiplication. Then find the centre of $GL_2(\mathbb{R})$. 5

5. Show that the set $\mathbb{Z}$, of all odd integers forms a group with respect to the binary operation $*$, where for $a, b \in \mathbb{Z}$, 5

$$a*b:=a+b-3$$

GROUP-C

Answer *all* the questions from the following 7×4= 28

6. Let $Q_8:=\left\{\begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} : a,\ b,\ c,\ d \in \mathbb{Z}\ ;\ \ a^2+b^2+c^2+d^2=1\ ;\ i^2=-1\right\}$ 7

Show that Q_8 forms a group with respect to matrix multiplication.

7. Let G be the open interval (–1, 1). Define the binary operation $*$ on G by 7

$$a*b \,:=\frac{a+b}{1+ab}\,, \quad \forall\, a,\, b\in G$$

Show that $(G,\,*)$ is a group.

8. (a) Let $\alpha=\begin{pmatrix}1&2&3&4\\4&3&1&2\end{pmatrix}$, $\beta=\begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix}$, $\gamma=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}$ in S_4. Compute each of the following: 4

(i) $\alpha\beta\gamma^{-1}$ (ii) $\alpha^2\beta$

(b) Prove that a group $(G,\,*)$ is commutative if $(a*b)^n=a^n*b^n$ for any three consecutive integers n and $\forall\, a,\, b\in G$. 3

9. (a) Check whether the set of residue classes modulo 5 forms a group with respect to multiplication of residue classes. 4

(b) Let G be a group and H be a non-empty finite subset of G. Then H is a subgroup of G iff for any two elements $a,\, b\in H$, $ab\in H$. 3

Given an example to show that the above result may not hold when H is infinite.

GE5

NUMERICAL METHODS

GROUP-A

1. Answer ***all*** the following questions: 2×6 = 12

(a) Round the numbers 0.01201 and -239.85 upto three significant digits.

(b) Locate the real root of $x^2+\ln x=2$ by graphical method.

(c) Give the geometrical significance of $f'(x)$ in approximation of the root of the equation $f(x)=0$ by Newton-Raphson method.

(d) Suggest a value of constant k so that the iteration formula $x=x+k(x^2-3)$ may converge at a good rate, given that $x=\sqrt{3}$ is a root.

(e) When a system of linear equations is said to be ill conditioned?

(f) Which one of the arrangements

(i) $x=(9x-1)^{1/3}$ (ii) $x=\dfrac{1}{9-x^2}$ (iii) $x=\sqrt{9-\dfrac{1}{x}}$

will give the root between 0 and 1 by fixed point iteration method of the equation $x^3-9x-1=0$? — Give reason.

GROUP-B

2. Answer ***all*** the following questions: 5×4 = 20

(a) Find a real root of the equation $x^3+x-3=0$ by Newton-Raphson method, correct upto 4-significant figures lying between 1.2 and 1.3.

(b) The fourth degree polynomial equation $f(x)=230x^4+18x^3+9x^2-221x-9=0$ has real root in $[-1,\ 0]$. Approximate that root upto two decimal places by Bisection method and that then upto four decimal places by the Secant method.

(c) Use fixed-point iteration method to find an approximation to $\sqrt[3]{25}$ that is accurate to within 10^{-4}. Compare your result and the number of iterations required with the answer as obtained by Bisection method.

(d) Use the method of iteration to find the root of the equation $10^x+x-4=0$ correct upto 5 decimal places.

GROUP-C

3. Answer ***all*** the following questions: 7×4 = 28

(a) Solve the equations using Gauss elimination method 5+2

$$0.34x_1-0.58x_2+0.94x_3=2$$
$$0.27x_1+0.42x_2+0.13x_3=1.5$$
$$0.2x_1-0.51x_2+0.54x_3=0.8$$

and show that the number of multiplication and divisions required to solve is 21.

(b) Reduce the following system of linear equations in the diagonally dominant system and then solve by Gauss-Seidel iteration method: 2+5

$$x_1+x_2+4x_3=9$$
$$8x_1-3x_2+2x_3=20$$
$$4x_1+11x_2-x_3=33$$

upto three significant figures.

(c) (i) The equation $x^2+ax+b=0$ has two real roots α, β. Show that the iteration method $x_{k+1}=-\dfrac{ax_k+b}{x_k}$ is convergent near α if $|\alpha|>|\beta|$. 4

(ii) Show that bisection method converges linearly. 3

(d) (i) Show that the iteration $x_{n+1}=\dfrac{(x_n+a/x_n)}{2}$ has a second order convergence to $\sqrt{a}$. 3

(ii) Solve the equations by an iterative method: 4

$$1.876x_1+2.985x_2-11.62x_3=-0.972$$
$$12.214x_1+2.367x_2+3.672x_3=7.814$$
$$2.412x_1+9.879x_2+1.564x_3=4.89$$

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